Improved Algorithm for Reachability in d-VASS

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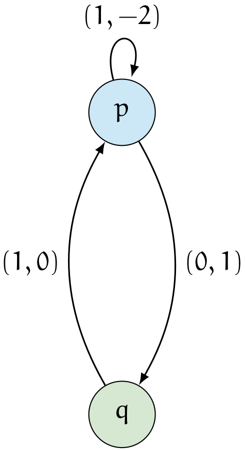
VASS and Reachability Problem

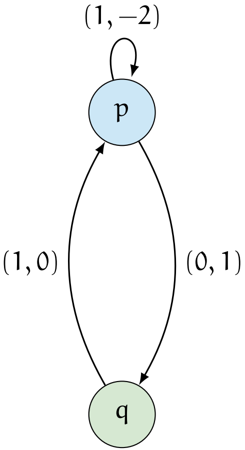
VASS



VASS = Vector Addition System with States

Finite state machine with transitions labeled by vectors.

VASS



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Definition

Formally, a d-dimensional VASS, or d-VASS for short, is a pair

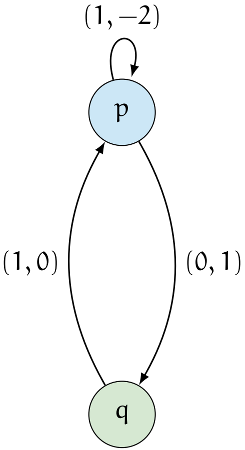
$$G = (Q, T)$$

$$T \subseteq Q \times \mathbb{Z}^d \times Q$$

effect

finite set of states

VASS



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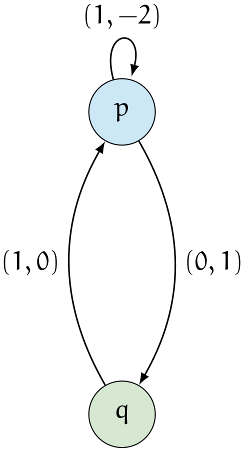
VASS = VAS = Petri Net = $\cdots$

Runs in VASS

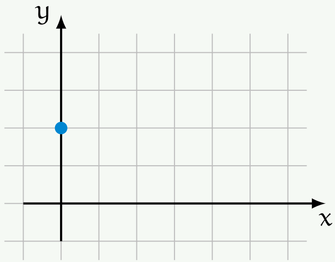


Configuration: $p(\vec{v}) \in Q \times \mathbb{N}^d$.

Move: $p(\vec{v}) \xrightarrow{(p,\vec{a},q)} q(\vec{v} + \vec{a})$.

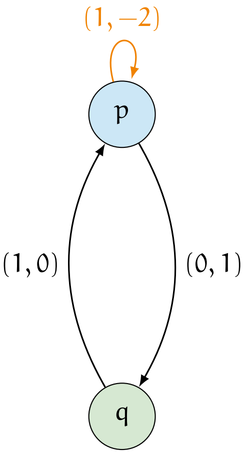
Rule: vectors at hand must be non-negative.

Example



$p(0, 2)$

Runs in VASS



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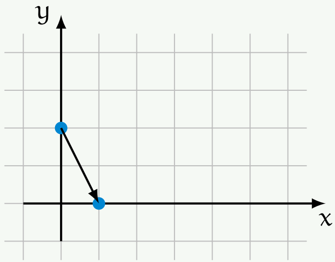

$p(1, 0)$

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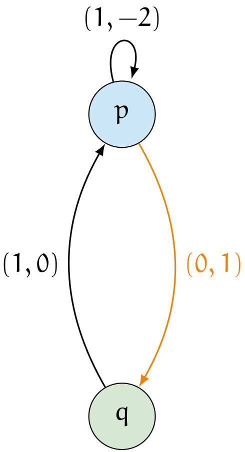

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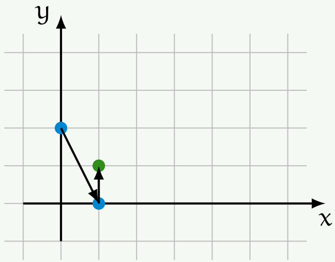

$q(1,1)$

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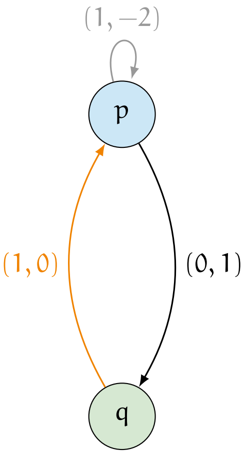

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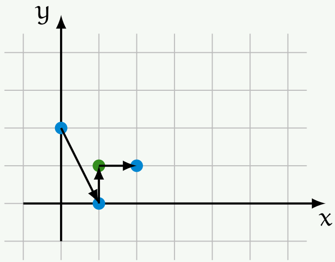
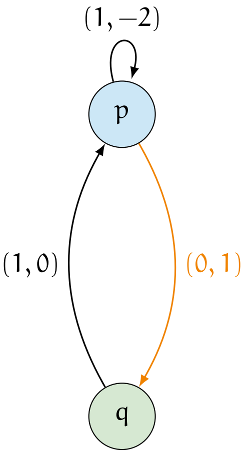


$p(2, 1)$



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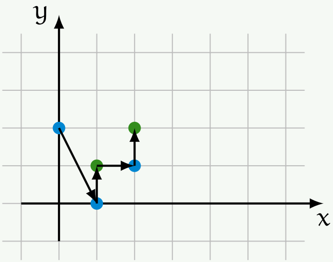

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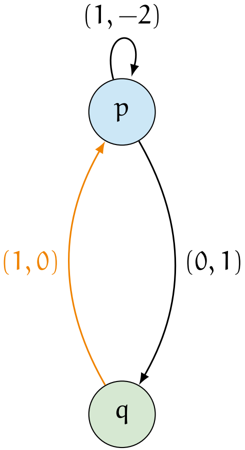

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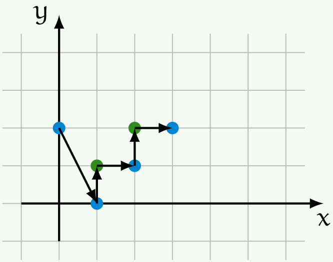
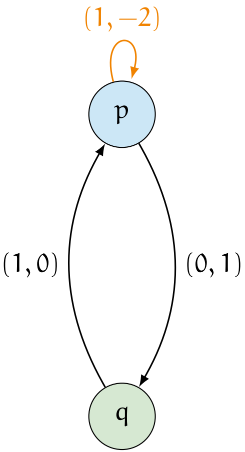


$p(3, 2)$

Runs in VASS



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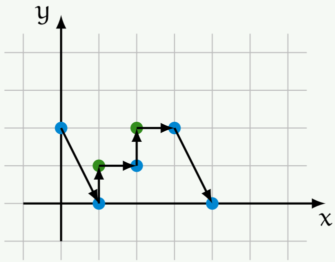

$p(4,0)$

Reachability problem

VASS Reachability Problem

Input: a VASS G and two configurations $p(\vec{x})$, $q(\vec{y})$.

Question: does $p(\vec{x}) \xrightarrow{*} q(\vec{y})$?

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Gap not closed between complexity upper and lower bounds.

History (upper bounds)

1969 Introduction of vector addition systems.
Karp, Miller. *Parallel program schemata.*

1977 (Incomplete) decidability proof.
Sacerdote, Tenney. *The decidability of the reachability problem for vector addition systems.*

1981 Decidability proof.
Mayr. *An algorithm for the general Petri net reachability problem.*

1982 Decidability proof, simplified.
Kosaraju. *Decidability of reachability in vector addition systems.*

1992 Decidability proof, refined.
Lambert. *A structure to decide reachability in Petri nets.*

2015 First upper bound: $\mathbf{F}_{\omega^3}$ (cubic-Ackermann).
Leroux, Schmitz. *Demystifying Reachability in Vector Addition Systems.*

2019 Improved upper bound: $\mathbf{F}_{\omega}$ (Ackermann) for general problem, $\mathbf{F}_{d+4}$ for d-VASS.
Leroux, Schmitz. *Reachability in Vector Addition Systems is Primitive-Recursive in Fixed Dimension.*

History (lower bounds)

1976 — EXPSPACE hardness.
Lipton. *The Reachability Problem Requires Exponential Space.*

2019 — $\mathbf{F}_3$(TOWER) hardness.
Czerwinski, Lasota, Lazic, Leroux, Mazowiecki. *The Reachability Problem for Petri Nets is Not Elementary.*

2021 — $\mathbf{F}_\omega$(Ackermann) hardness for general problem, F_d-hardness for $6d$-VASS.
Czerwinski, Orlikowski. *Reachability in Vector Addition Systems is Ackermann-complete.*

2021 — $\mathbf{F}_\omega$(Ackermann) hardness for general problem, F_d-hardness for $(4d+9)$-VASS.
Leroux. *The Reachability Problem for Petri Nets is Not Primitive Recursive.*

2023 — $\mathbf{F}_\omega$(Ackermann) hardness for general problem, F_d-hardness for $(2d+3)$-VASS.
Czerwinski, Jecker, Lasota, Leroux, Orlikowski. *New Lower Bounds for Reachability in Vector Addition Systems.*

Our results

Theorem [Leroux and Schmitz, 2019]

Reachability for $\mathbf{d}$-VASS is in $\mathbf{F}_{\mathbf{d}+4}$.

$$\Downarrow$$

Theorem [Ours]

Reachability for $\mathbf{d}$-VASS is in $\mathbf{F}_{\mathbf{d}}$.

What is $\mathbf{F}_{\mathfrak{d}}$?

Complexity classes defined in [Schmitz. *Complexity Hierarchies Beyond Elementary.* 2016.]

What is $\mathbf{F}_d$?

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Fast-growing Functions

$$
\begin{aligned}
F_0(x) &= x+1\\
F_1(x) &= \underbrace{F_0(F_0(\cdots(F_0}_{x+1\text{ times}}(x))\cdots)) = 2x+1\\
F_2(x) &= F_2^{x+1}(x) = 2^x(x+1)-1\\
&\ \vdots\\
F_d(x) &= F_{d-1}^{x+1}(x)\\
&\ \vdots\\
&\text{tends to the Ackermannian function}
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The $\mathscr{F}_d$ Hierarchy

$$\mathscr{F}_d := \bigcup_{c \in \mathbb{N}} \mathsf{FDTime}(F_d^c(n))$$

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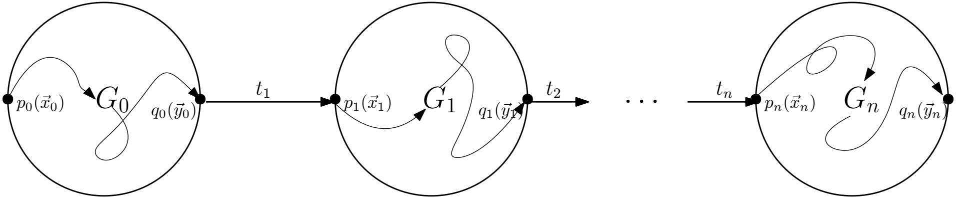
$$\mathbf{F}_d := \bigcup_{p \in \mathscr{F}_{d-1}} \mathsf{DTime}(F_d(p(n)))$$

E.g., $F_3 =$ Tower is already non-elementary.

All F_d for $d \in \mathbb{N}$ are primitive recursive.

Technical Tool I: KLMST Decomposition

Decomposition Structure



KLM Sequence: a sequence of (generalized) reachability instances linked by transitions.

"generalized": some components in the vectors $\vec{x}_i, \vec{y}_i$ can be missing

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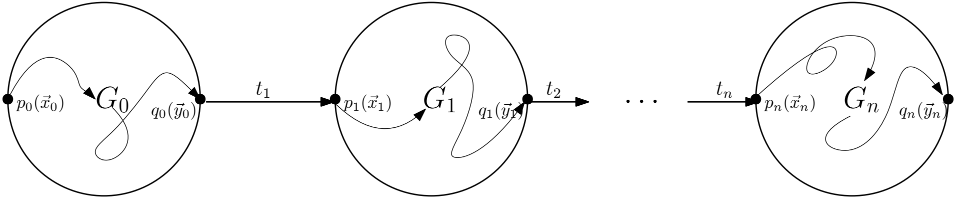

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To split a question of the form "is $q(\vec{y})$ reachable from $p(\vec{x})$ in G" into consecutive easier questions.

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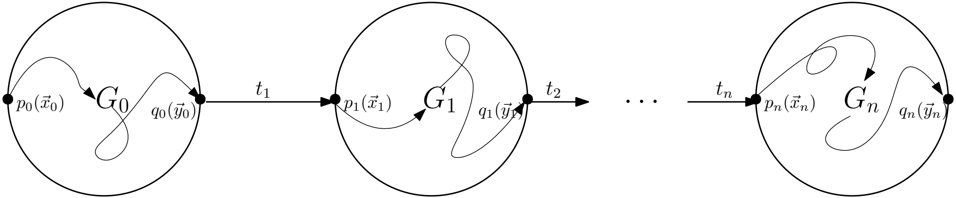

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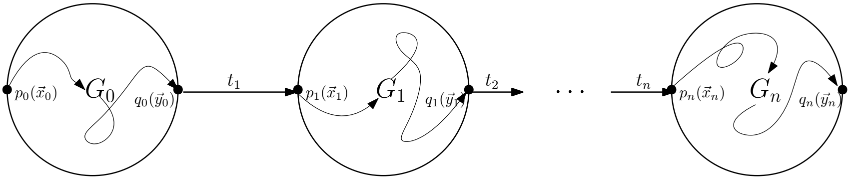
Initially, we start with a single reachability instance: the input instance.

Normal conditions



Definition

A KLM sequence is normal if it is

solvable

strongly connected

saturated

unbounded

rigid

pumpable

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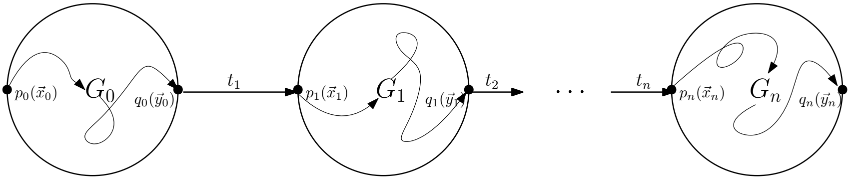

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Theorem [Leroux and Schmitz, 2019]

If a KLM sequence is normal, then there is a run from the very beginning to the very end.

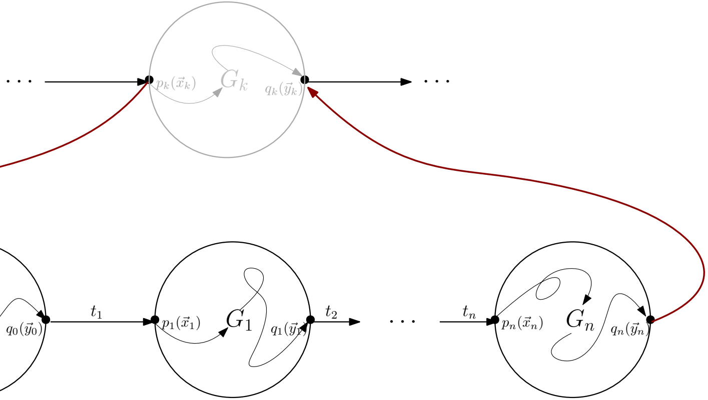
Fact [Leroux and Schmitz, 2019]

Whether a KLM sequence is normal is decidable in EXPSPACE.

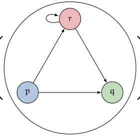
Decomposition

Theorem [very informal and inaccurate!]

If a KLM sequnece ξ is not normal, then one can compute a set Ξ of KLM sequences of smaller ranks, such that each run admitted by ξ is admitted by some sequence in Ξ, and every run admitted by Ξ is admitted by ξ, i.e. reachability witnesses are preserved exactly.



Ranks

rank() =

$$\begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_i \\ \vdots \\ r_d \end{pmatrix}$$

where $r_i = \#$ transitions t
such that the vector space spanned by the effect
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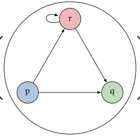
Ranks

rank() $= \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_i \\ \vdots \\ r_d \end{pmatrix}$

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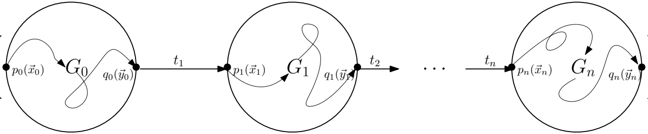
rank()

$= \text{rank}(G_0) + \text{rank}(G_1) + \cdots + \text{rank}(G_n)$

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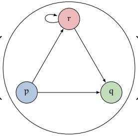
rank(
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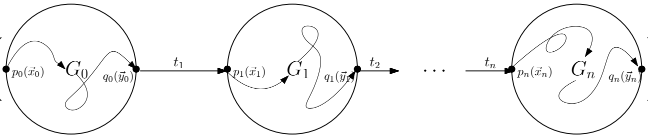
rank(
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$$= \mathrm{rank}(G_0) + \mathrm{rank}(G_1) + \cdots + \mathrm{rank}(G_n)$$

Ranks are ordered lexicographically.

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Complexity: $\mathbf{F}_{d+4}$ by [Leroux and Schmitz, 2019], where

$$d+4 = \begin{array}{cl} d+1 & \longleftarrow \text{length of ranking function} \\ + & \\ 3 & \longleftarrow \text{elementary } (< F_3) \text{ blow up in each decomposition step} \end{array}$$

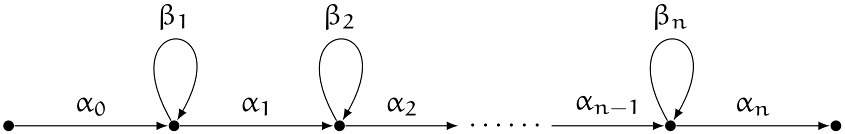
Technical Tool II: Linear Path Schemes

Linear path schemes

A linear path scheme for a VASS G characterize a set of paths in G by a regular expression

$$\alpha_0\ \beta_1^+\ \alpha_1\ \beta_2^+\ \alpha_2\ \ldots\ \alpha_{n-1}\ \beta_n^+\ \alpha_n$$

where $\alpha_0, \ldots, \alpha_n$ are paths and $\beta_1, \ldots, \beta_n$ are cycles in G.



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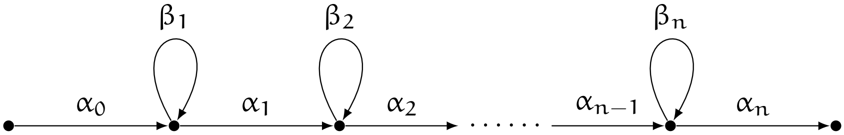

A path π is admitted by this linear path scheme if it can be written as

$$\pi = \alpha_0 \; \beta_1^{e_1} \; \alpha_1 \; \beta_2^{e_2} \; \alpha_2 \; \ldots \; \alpha_{n-1} \; \beta_n^{e_n} \; \alpha_n$$

where $e_1, \ldots, e_n > 0$.

Characterization systems

Runs admitted by a linear path scheme are **completely** characterized by a system of linear inequalities.

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To decide whether there is a run of the form

$$\begin{tikzcd}
p(\vec{x}) \arrow[r, "\alpha_0"] & \bullet \arrow[loop above, "\beta_1^{e_1}"] \arrow[r, "\alpha_1"] & \bullet \arrow[loop above, "\beta_2^{e_2}"] \arrow[r, "\alpha_2"] & \cdots\cdots \arrow[r, "\alpha_{n-1}"] & \bullet \arrow[loop above, "\beta_n^{e_n}"] \arrow[r, "\alpha_n"] & q(\vec{y})
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we only need to check if $e_1, e_2, \ldots, e_n$ satisfies the following (linear) constraints:

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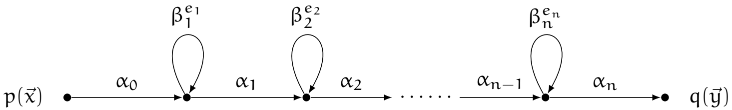

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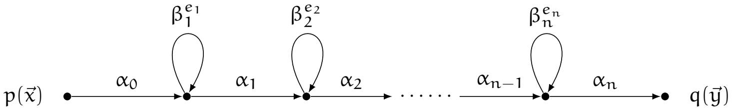

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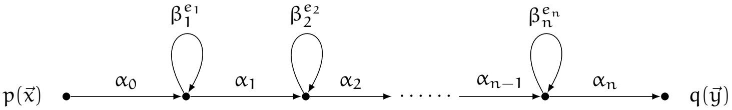

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- the total effect is $\vec{y} - \vec{x}$,
- each configuration along each path segment α_i is $\geqslant \vec{0}$,
- each configuration along the first and last lap of each cycle segment β_i is $\geqslant \vec{0}$,

Characterization systems

> Runs admitted by a linear path scheme are **completely** characterized by a system of linear inequalities.

To decide whether there is a run of the form

$$\begin{tikzcd} p(\vec{x})\ \bullet \arrow[r, "\alpha_0"] & \bullet \arrow[loop above, "\beta_1^{e_1}"] \arrow[r, "\alpha_1"] & \bullet \arrow[loop above, "\beta_2^{e_2}"] \arrow[r, "\alpha_2"] & \cdots\cdots \arrow[r, "\alpha_{n-1}"] & \bullet \arrow[loop above, "\beta_n^{e_n}"] \arrow[r, "\alpha_n"] & \bullet\ q(\vec{y}) \end{tikzcd}$$

we only need to check if $e_1, e_2, \ldots, e_n$ satisfies the following (linear) constraints:

- the total effect is $\vec{y} - \vec{x}$,
- each configuration along each path segment α_i is $\geqslant \vec{0}$,
- each configuration along the first and last lap of each cycle segment β_i is $\geqslant \vec{0}$,
- each $e_i \geqslant 1$.

Application in 2-VASS

Theorem [Blondin, Englert, Finkel, Göller, Hasse, Lazic, McKenzie, Totzke, 2021]

For any 2-dimensional VASS G, there is a finite set of linear path schemes of length $\mathrm{poly}(|G|)$ such that any run in G is admitted by some linear path schemes in this set.

As a corollary, reachability in 2-VASS is shown to be in PSPACE.

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However, the theorem fails already for 3-VASS.

Question: Can we apply the linear path scheme technique to KLMST decomposition?

Geometrically 2-dimensional VASS

Geometrical Dimension of a VASS

Definition

The geometrical dimension of a VASS G is the dimension of the vector space spanned by the effects of all (simple) cycles in G.

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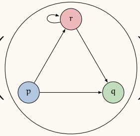
Example

$$\mathrm{gdim}\left(\begin{tikzcd} p \arrow[loop left, "{(1,0,-1)}"] \arrow[loop right, "{(0,1,1)}"] \end{tikzcd}\right) = 2$$

Low rank $\implies$ low geometrical dimension

Recall

$$\operatorname{rank}\left(\right.$$  $$\left.\right) = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_i \\ \vdots \\ r_d \end{pmatrix}$$

where $r_i = \#$ transitions t such that the vector space spanned by the effect of cycles containing t has dimension i

Lemma

If G is a strongly connected d-VASS whose rank is $(r_0, \ldots, r_d)$ such that $r_3 = r_4 = \cdots = r_d = 0$, then $\mathrm{gdim}(G) \leqslant 2$, i.e., G is a geometrically 2-dimenisonal VASS.

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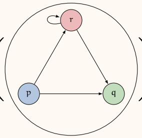

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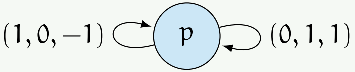
Since the ranks of KLM sequences keep dropping during the KLMST decomposition, one will eventually obtain geometrically 2-dimenisonal VASSes.

Geometrically 2-VASS ≈ 2-VASS

Idea: project a geometrically 2-VASS onto 2 coordinates to get a 2-VASS.



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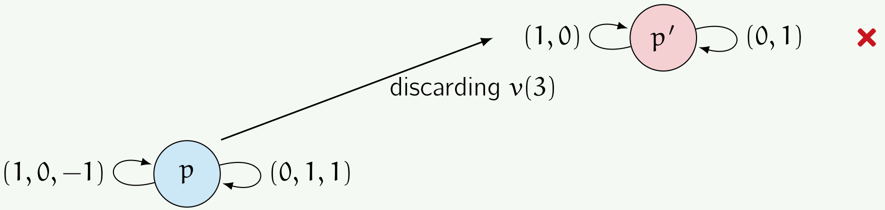

Geometrically 2-VASS $\approx$ 2-VASS

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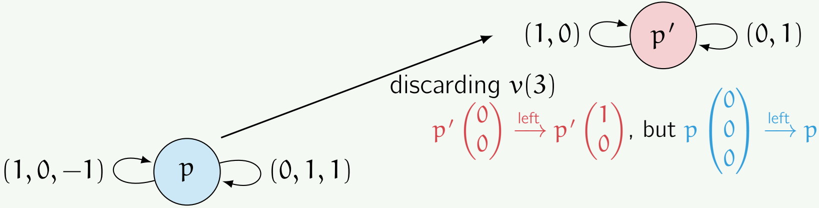

$p' \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\text{left}} p' \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, but $p \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{left}} p \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

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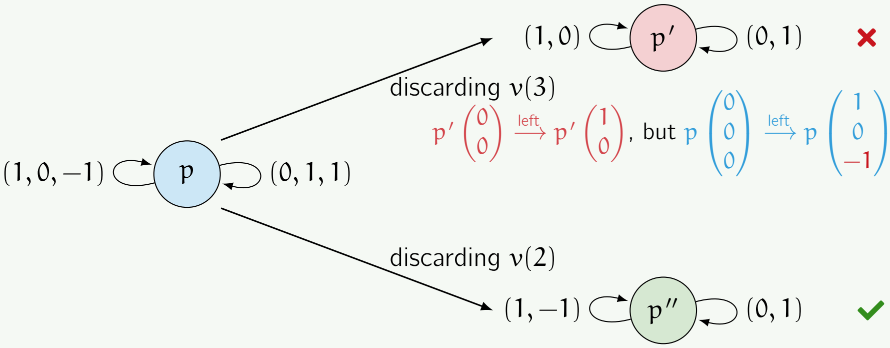

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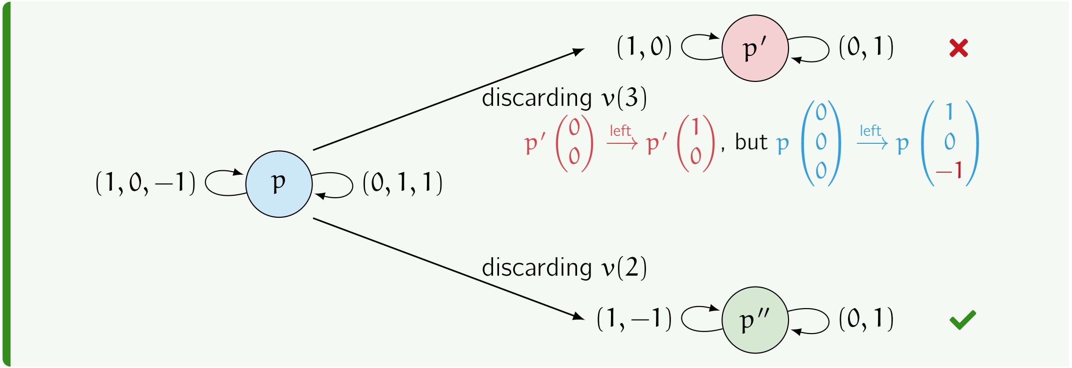


Every run from $p''(0,0)$ can be safely project back to a valid run from $p(0,0,0)$:

$$\forall\, \vec{v} \in \text{span}\left\{ \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}, \quad \vec{v}(1) \geqslant 0 \land \vec{v}(3) \geqslant 0 \implies \vec{v} \geqslant \vec{0}$$

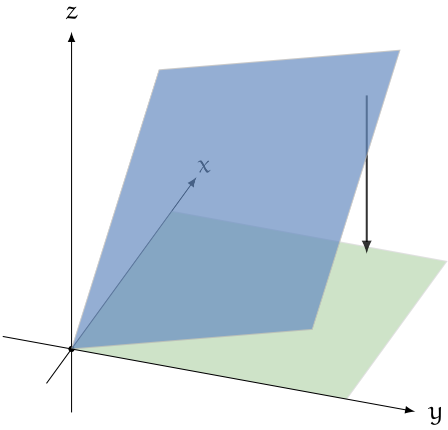
Sign-reflecting projection

Lemma

Every (2-dimensional) plane (satisfying some regular conditions) in $\mathbb{Q}^d$ can be projected onto a coordinate plane such that

$$\text{projected vector} \geqslant \vec{0} \iff \text{original vector} \geqslant \vec{0}$$

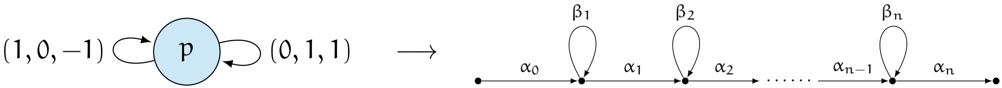
Such a projection is called a sign-reflecting projection.



Linear path schemes for geometrically 2-VASS

Theorem

For any geometrically 2-dimensional d-VASS G, there is a finite set of linear path schemes of length poly($|G|$) such that any run in G is admitted by some linear path schemes in it.



Modified KLMST Algorithm

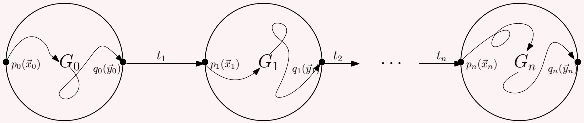
Basic idea

Replace every geometrically 2-VASS in a KLM sequence by linear path schemes.

Linear KLM Sequence

KLM sequence

A sequence of generalized reachability instances linked by transitions:



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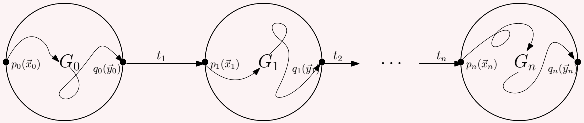

$\Downarrow$

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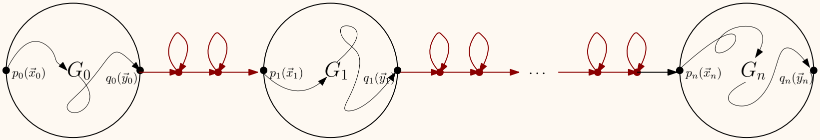

New normal conditions

Definition

A linear KLM sequence is *pure* if every geometrically 2-dimensional VASS in it is trivial (contains only a single state).

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Definition

A linear KLM sequence is normal if it is

solvable

strongly connected

saturated

unbounded

rigid

pumpable

pure

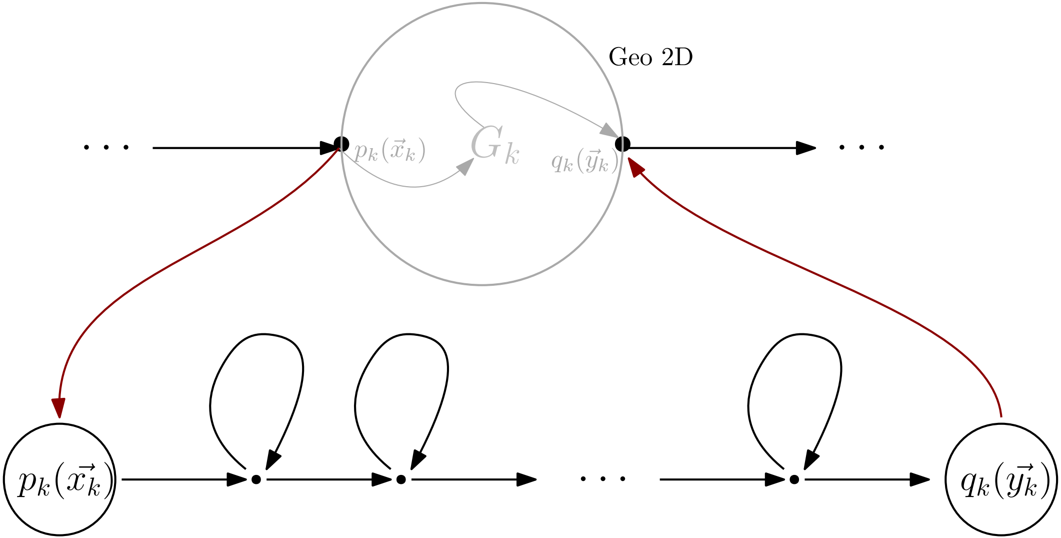
Theorem

If a linear KLM sequence is normal, then there is a run from the very beginning to the very end.

Fact

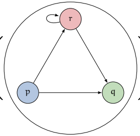
Whether a linear KLM sequence is normal is decidable in EXPSPACE.

New decomposition operation



New ranking function

We can now ignore the lower 3 components r_0, r_1, r_2 in the ranking function:

$\overline{\text{rank}}\Bigg($ 

$\Bigg)$ $:= \begin{pmatrix} r_3 \\ r_4 \\ \vdots \\ r_i \\ \vdots \\ r_d \end{pmatrix} \in \mathbb{N}^{d-2}$

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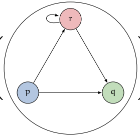

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Still guarantee that $\overline{\text{rank}}$ drops after each decomposition step.

Modified KLMST Decomposition Algorithm

Algorithm (same as before)

- Start with the initial linear KLM sequence which is the input instance.
- While the current linear KLM sequence is not normal, perform a decomposition to it and continue with any resulting sequence chosen non-deterministically.
- As long as one non-deterministic branch ends up with a normal sequence, we report "REACHABLE".

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Complexity

$$\mathsf{F}_{d+1} \quad \text{where } d+1 = \begin{array}{ll} d-2 & \longleftarrow \text{length of new ranking function} \\ + & \\ 3 & \longleftarrow \text{elementary } (< F_3) \text{ blow up in each decomposition step} \end{array}$$

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Complexity, with careful analysis on fast-growing functions

$$F_d \quad \text{where } d = \underbrace{d-2}_{\text{length of new ranking function}} + \underbrace{2}_{\text{elementary } (< F_2^c) \text{ blow up in each decomposition step}}$$

$d - 2 \longleftarrow$ length of new ranking function

$2 \longleftarrow$ elementary ($< F_2^c$) blow up in each decomposition step

Summary

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Reachability for d-VASS is in $\mathbf{F}_d$.

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Future directions

lower bound		upper bound
$\mathbf{F}_{O(d/2)}$	—— GAP ——	$\mathbf{F}_d$

Improved Algorithm for Reachability in d-VASS

Yuxi Fu[1], Qizhe Yang[2], **Yangluo Zheng**[1]

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